Day 06

Horn's Method and Fiducial Registration Error

Issues

- the use of the least-squares criteria assumes
  - identically distributed noise in each point
  - isotropic noise in each point
    - more accurate (although more complicated) algorithms are available if these criteria are not met
      - Matei and Meer, IEEE PAMI, 28(10), Oct 2006

Characterizing Error

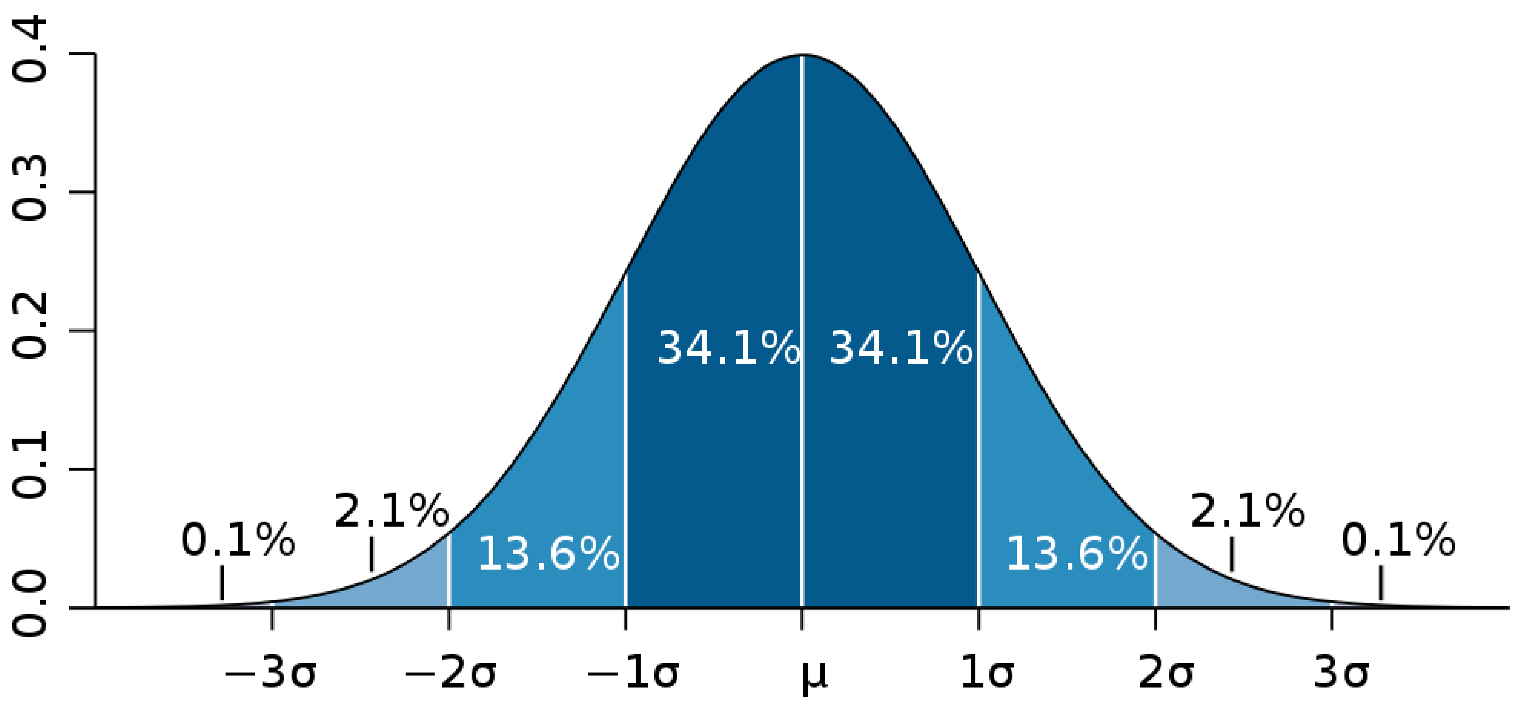
- most commonly used noise distribution is the zero-mean Gaussian (or normal) distribution

$$\mathrm{FLE}_i \sim \mathcal{N}(\mu, \Sigma)$$

- μ mean (location)
- Σ covariance (spread)

Characterizing Error

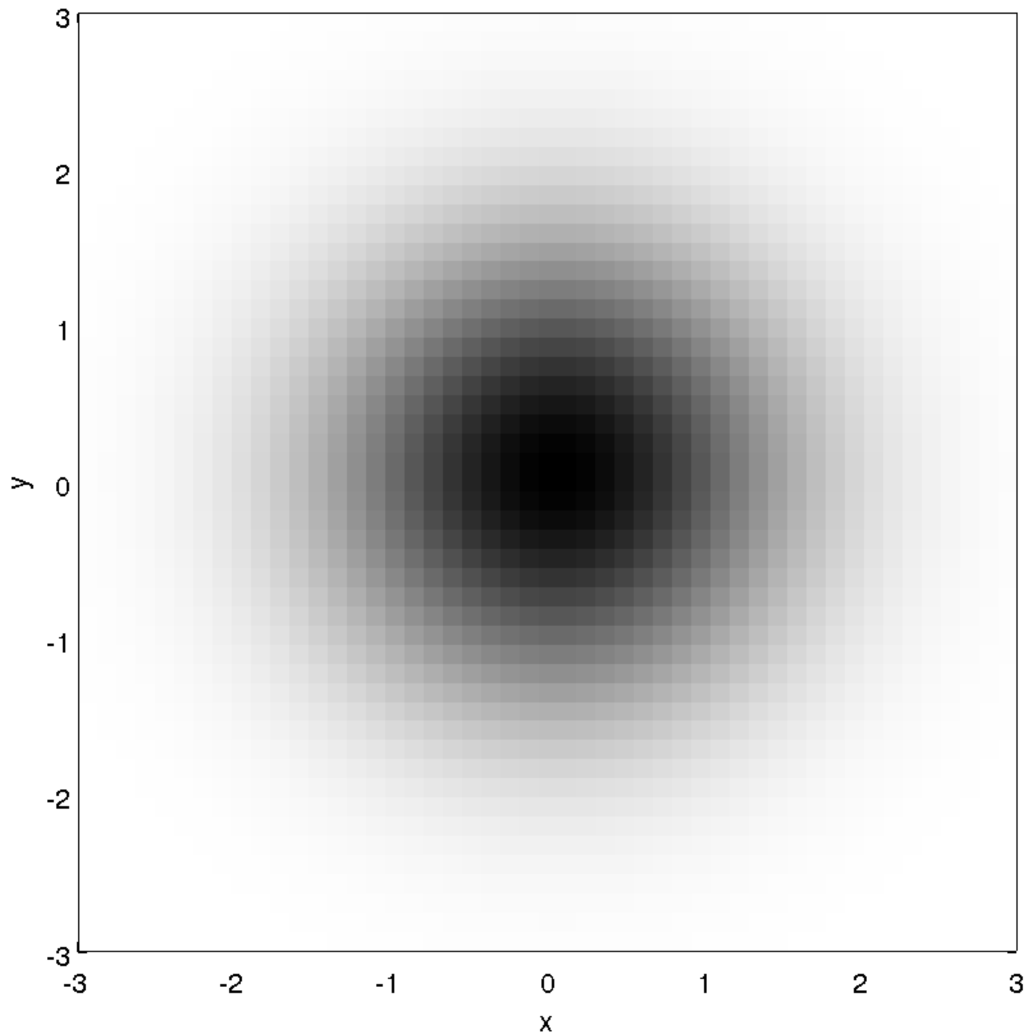
- in $1D$
  - σ standard deviation
  - $\Sigma = \sigma^2$ variance



Characterizing Error

- in $2D$
  - isotropic

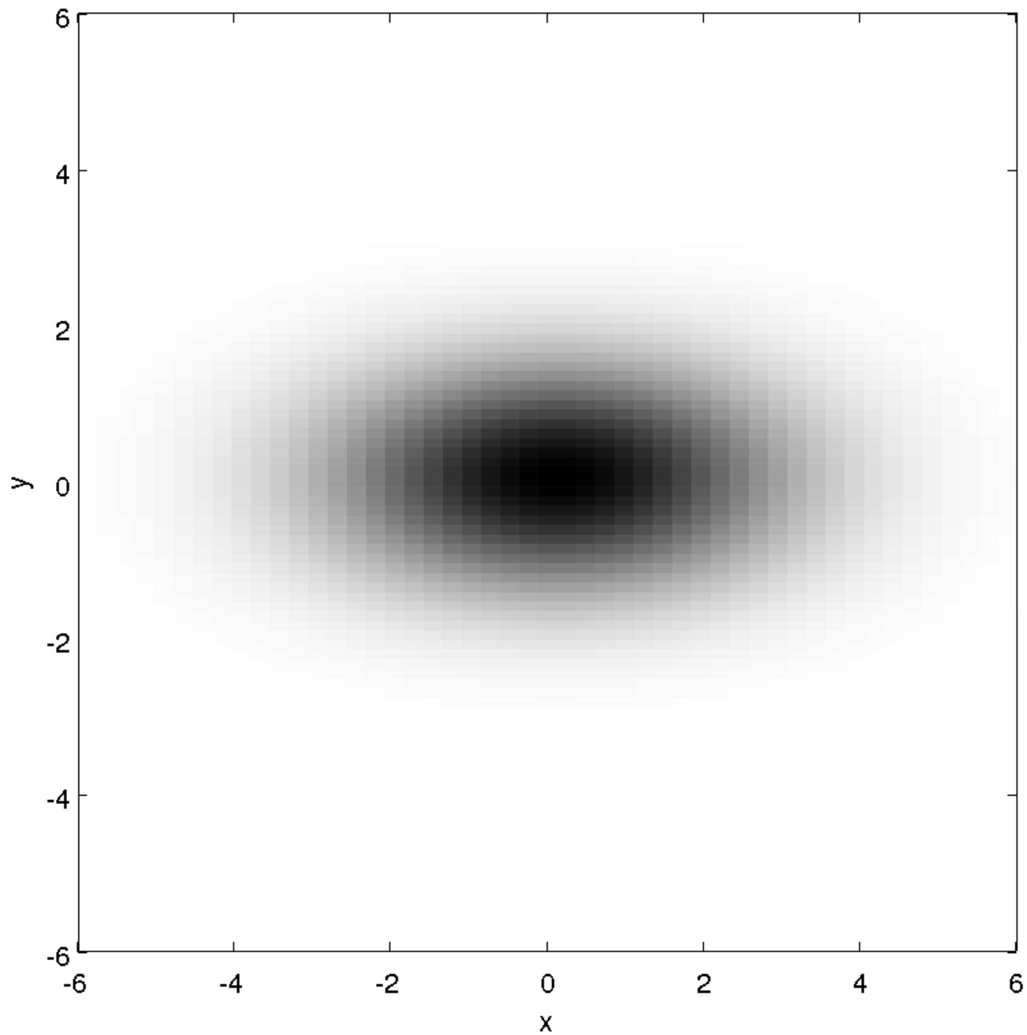
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Characterizing Error

- in $2D$
  - anisotropic

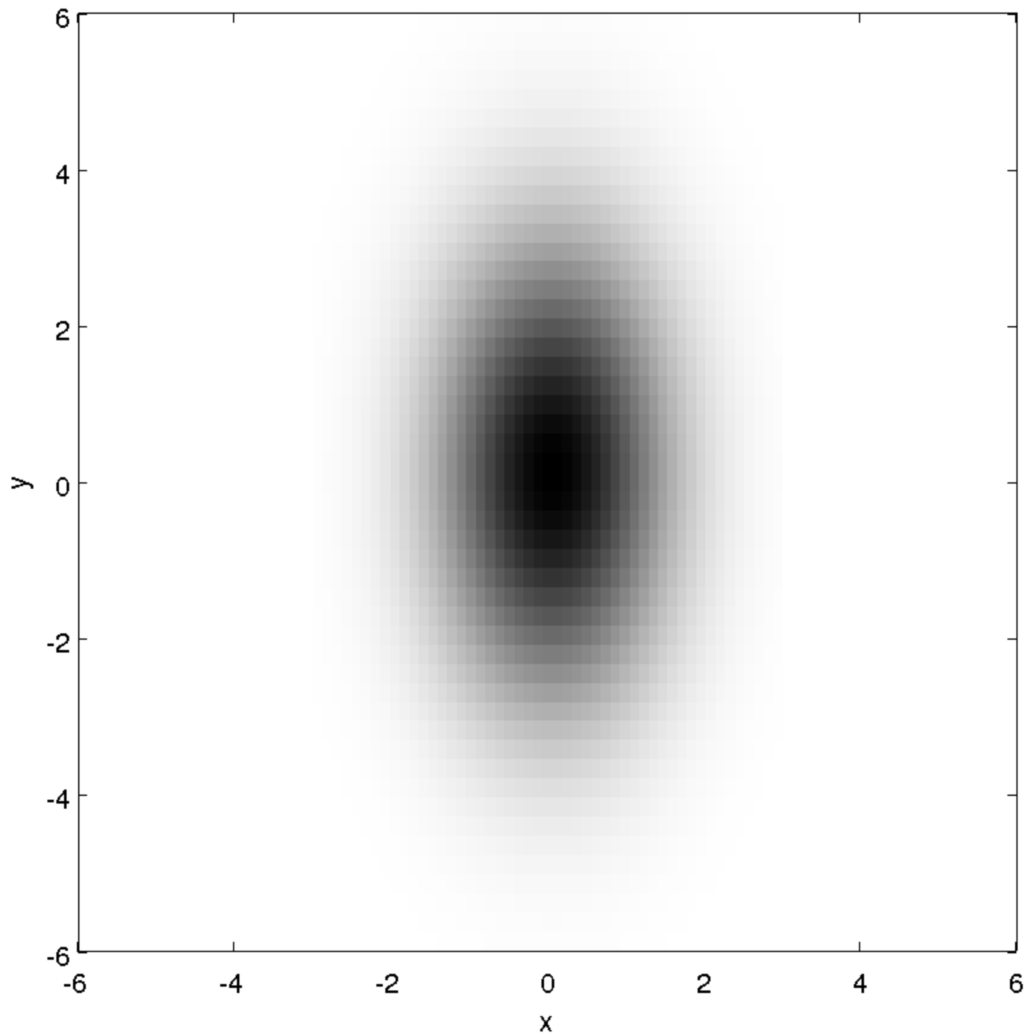
$$\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$



Characterizing Error

- in $2D$
  - anisotropic

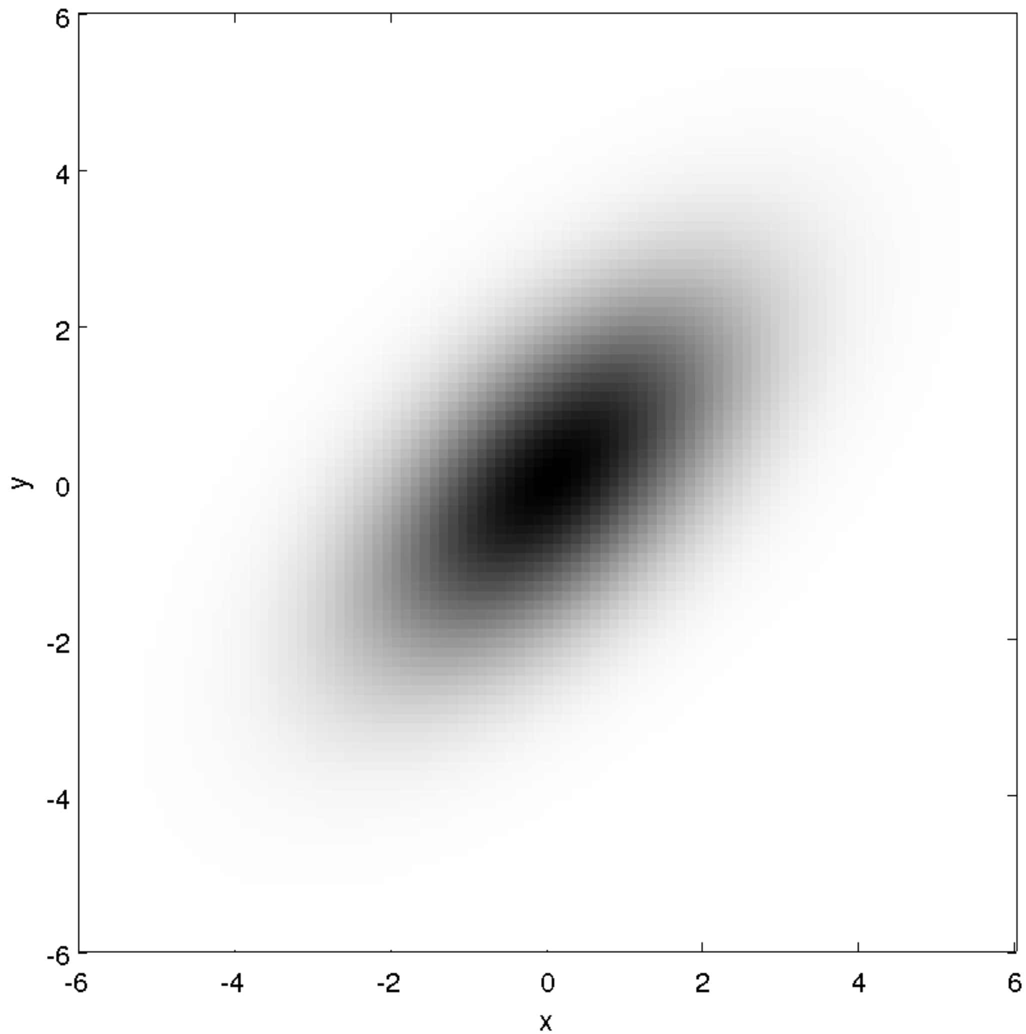
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$



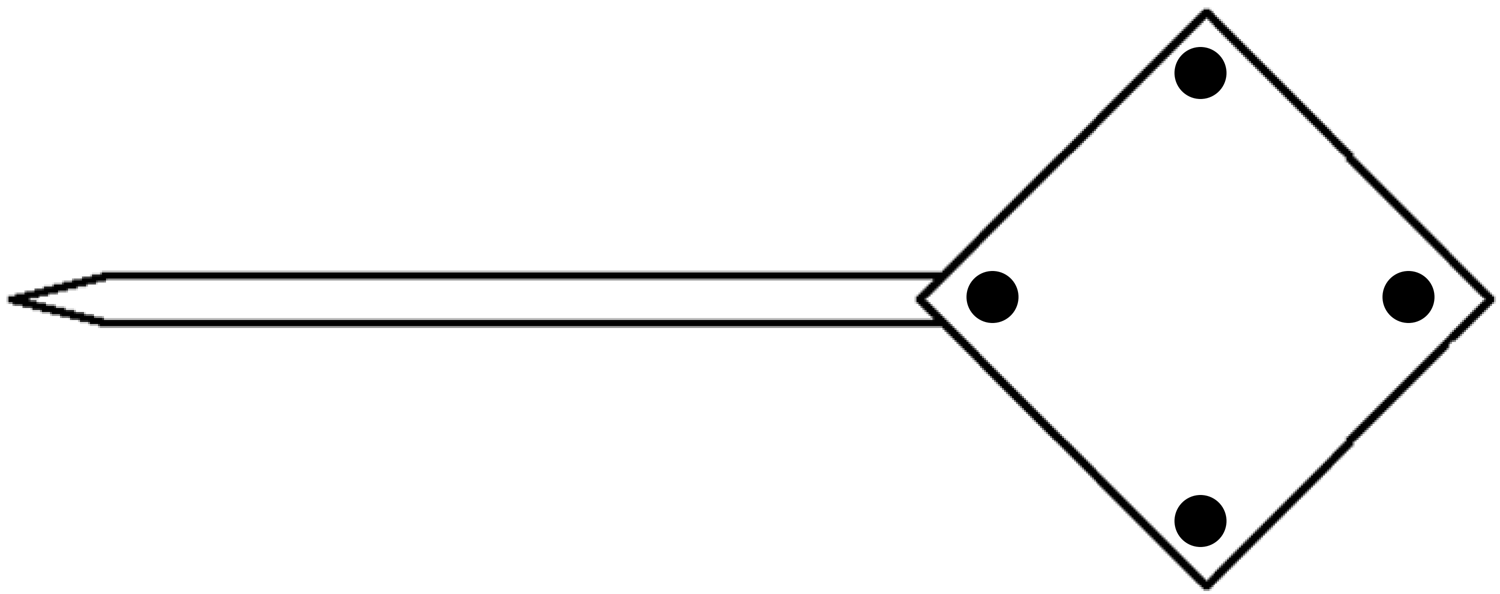
Characterizing Error

- in $2D$
  - anisotropic

$$\Sigma = \begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix}$$

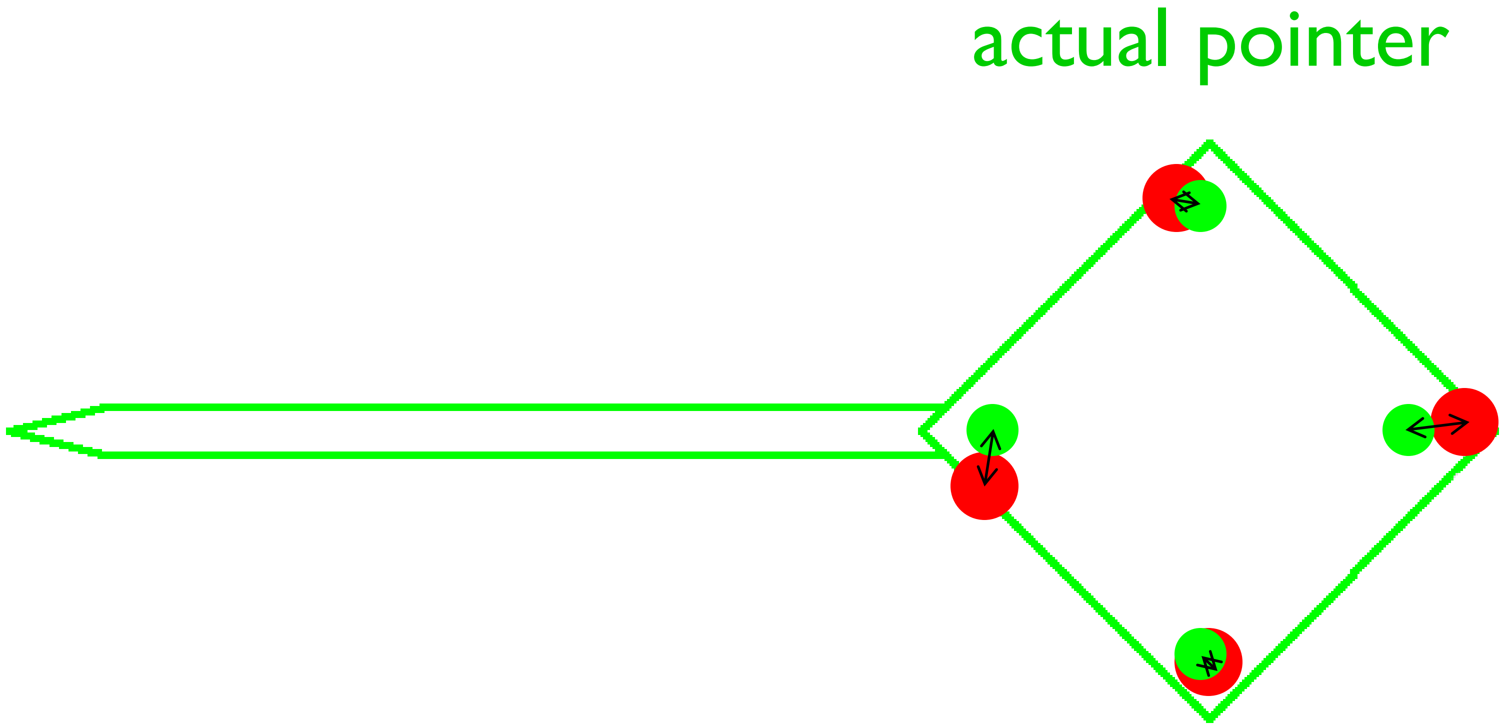


Registration Error



Suppose we have a tracked pointing stylus with a DRB having 4 fiducial markers.

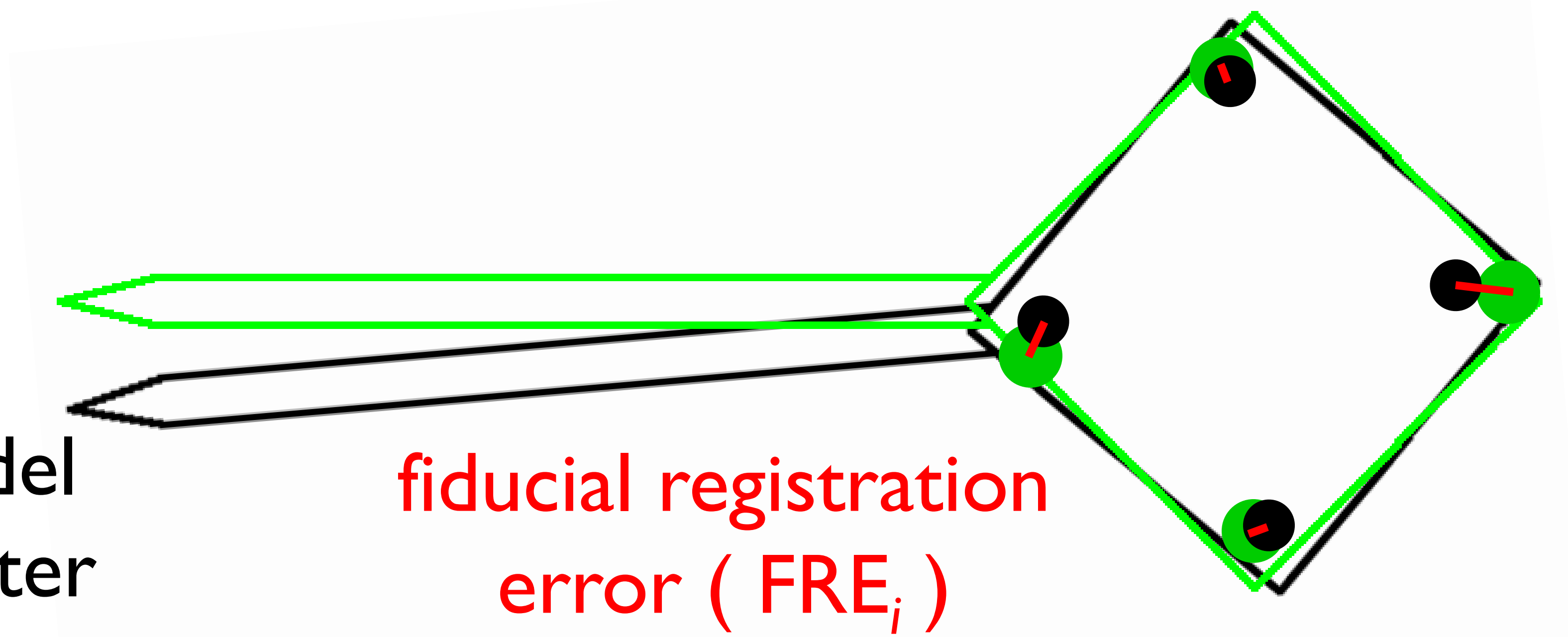
Registration Error



Because the of measurement errors in the tracking system, the locations of the fiducial markers cannot be measured exactly. The error between the actual and measured marker locations is called the fiducial localization error (FLE).

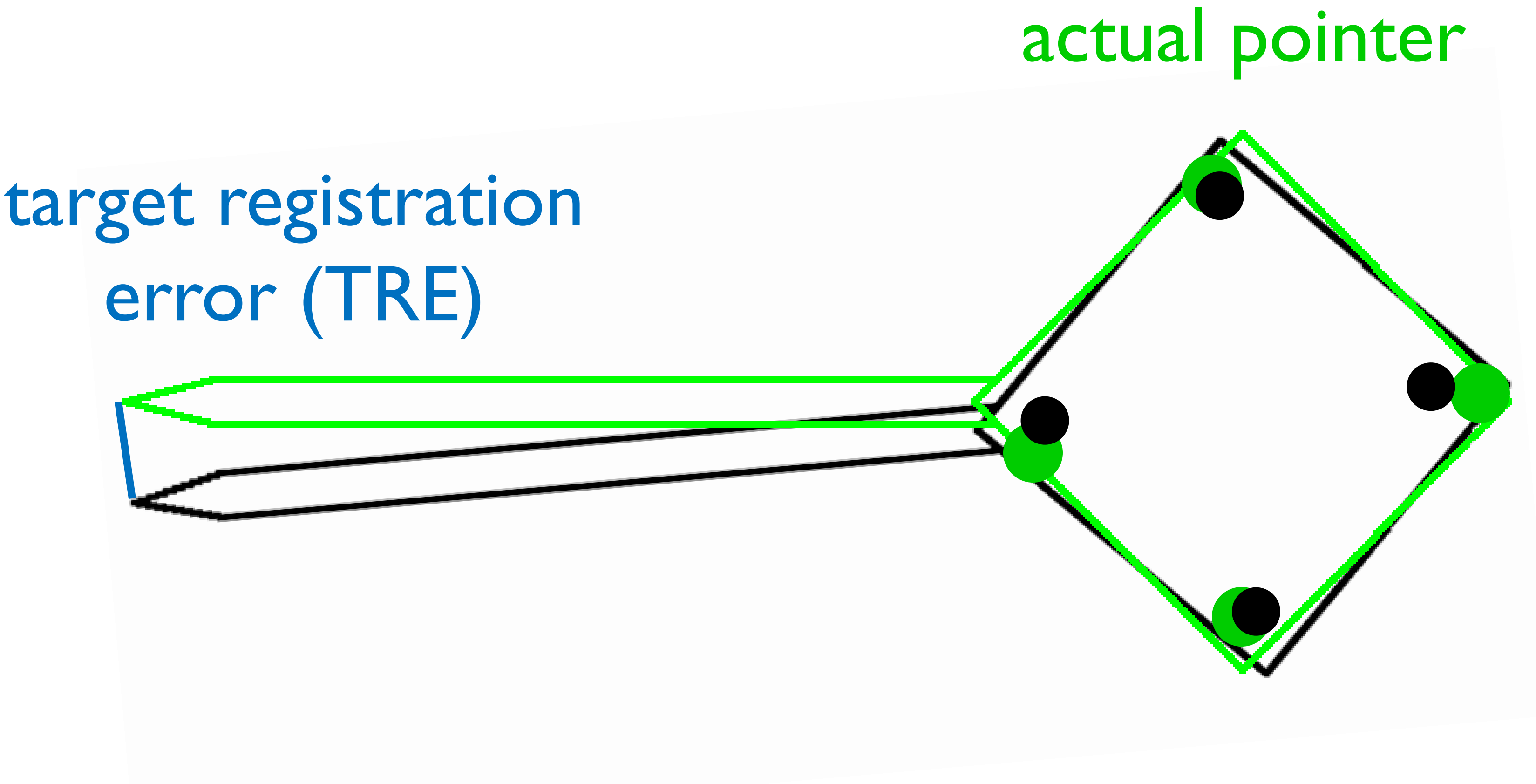
Registration Error

minimize : $\Sigma\ (\ FRE_i\ )^2$



When the model pointer is registered to the measured pointer, the FLE will lead to some error in the estimated rotation and translation. The residual errors in the fiducial locations after registration is called the fiducial registration error (FRE).

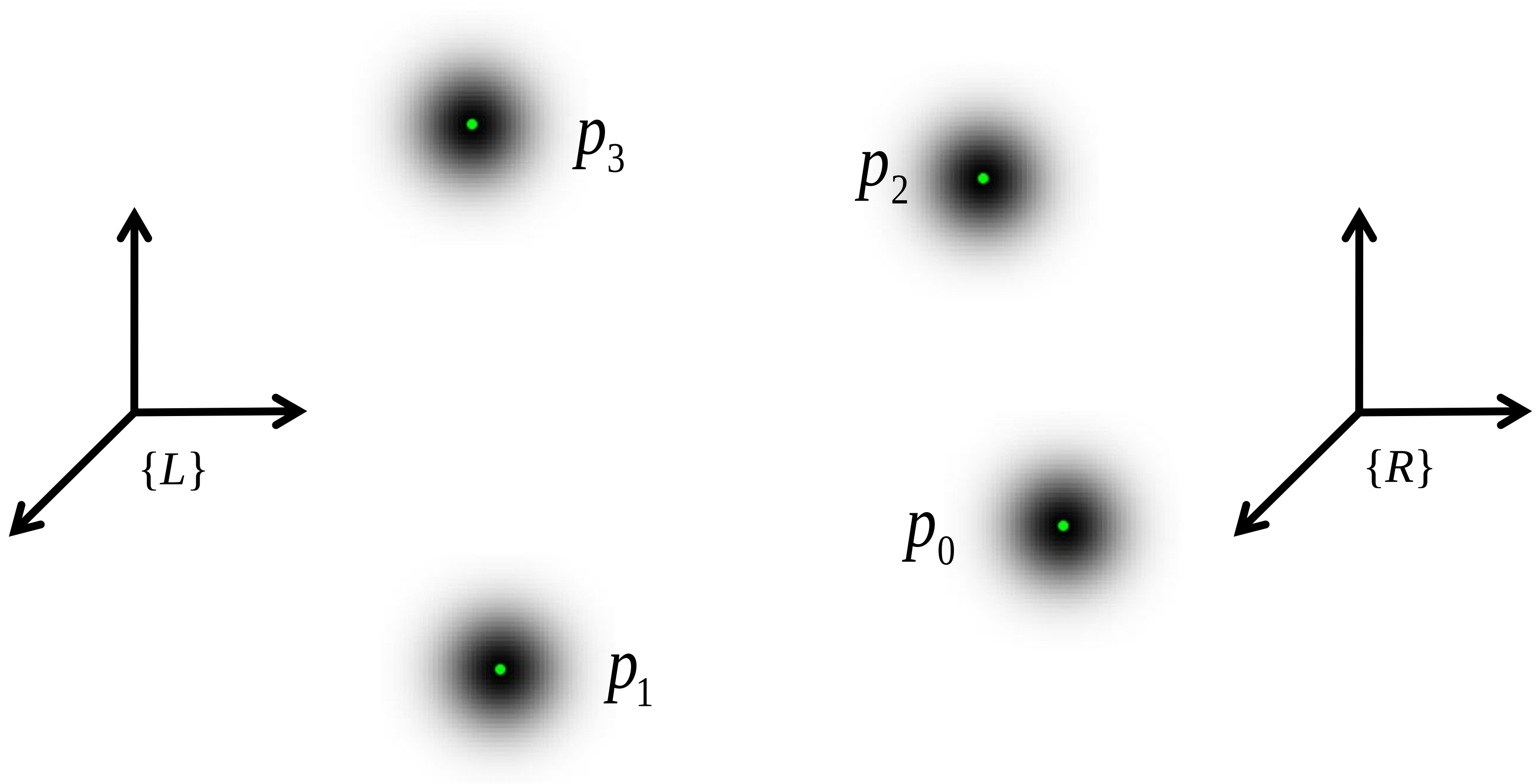
Registration Error



Usually, we are interested in points that are not fiducial locations. Any such point (not used for registration purposes) is called a target. The error between the true target position and registered target position is called the target registration error (TRE).

Horn's Method and FLE

- Horn's method (and all other ordinary least-squares methods) is optimal when FLE is identical and isotropic



Horn's Method and TRE

- early methods of studying the behaviour of TRE relied on simulation studies
  - IEEE Transactions on Medical Imaging, vol. 16, no. 4, Aug. 1997

Given a set of registration points $\{P\} = \{p_0, p_1, \ldots, p_{n\text{-}1}\}$, a target point t, and an FLE variance σ^2:

1. define noise

$$n_i = \mathcal{N}\left(\begin{bmatrix}0 & 0 & 0\end{bmatrix}^T, \begin{bmatrix}\sigma^2/3 & 0 & 0 \\ 0 & \sigma^2/3 & 0 \\ 0 & 0 & \sigma^2/3\end{bmatrix}\right)$$

Horn's Method and TRE

2. repeat 10,000 times
   - create a noise copy $\{Q\}$ of $\{P\}$ where $q_i = p_i + n_i$
   - register $\{Q\}$ to $\{P\}$ using Horn's method to obtain the rotation R_j and translation d_j
   - compute the registered target location

$$t_j = R_j\, t + d_j$$

   - compute the squared TRE

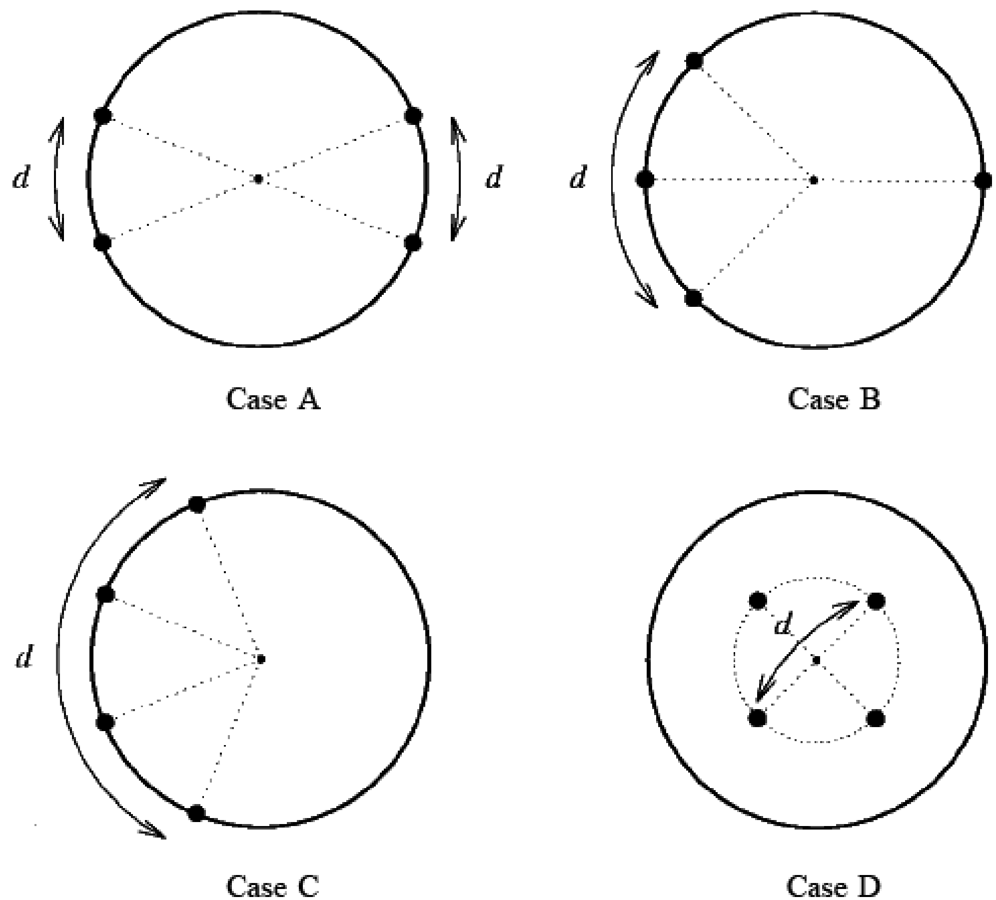
$$\mathrm{TRE}_{\mathrm{j}}^{2} = \left\| t_j - t \right\|^2$$

3. compute the root mean squared (RMS) TRE

$$\mathrm{TRE}_{\mathrm{RMS}} = \sqrt{\frac{1}{10{,}000} \sum_{j=1}^{10{,}000} \mathrm{TRE}_{\mathrm{j}}^{2}}$$

Horn's Method and TRE

- simulations performed for different configurations of markers



Horn's Method and TRE

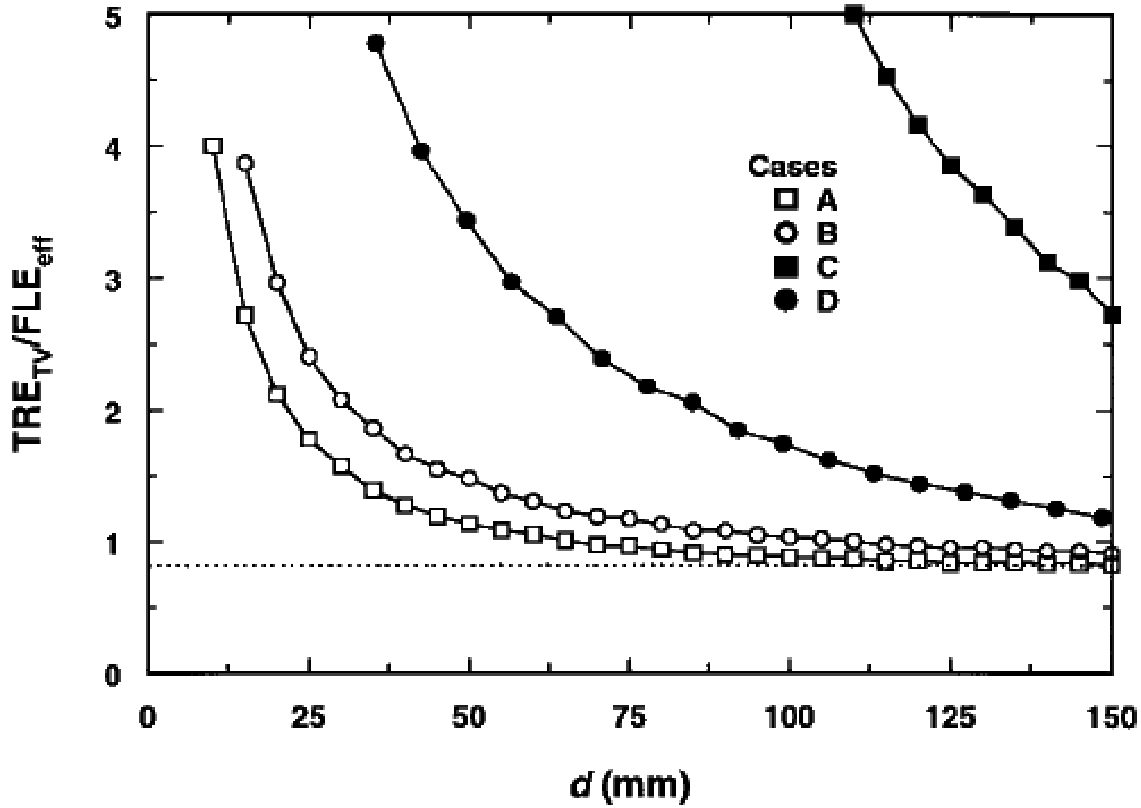


Fig. 11. Effect of distance between fiducials on TRE. Each symbol represents the mean theoretical $\mathrm{TRE}_{\mathrm{TV}}/\mathrm{FLE}_{\mathrm{eff}}$ predicted by numerical simulation for one of the four fiducial configurations shown in Fig. 10. The x-axis is the distance d in that figure. The dotted line is the mean $\mathrm{TRE}_{\mathrm{TV}}/\mathrm{FLE}_{\mathrm{eff}}$ when the four fiducials are distributed evenly around the circumference of the head (e.g., Case A, $d = 2\pi R/4 = 157$ mm).

Horn's Method and TRE

- simulation of TRE vs FLE

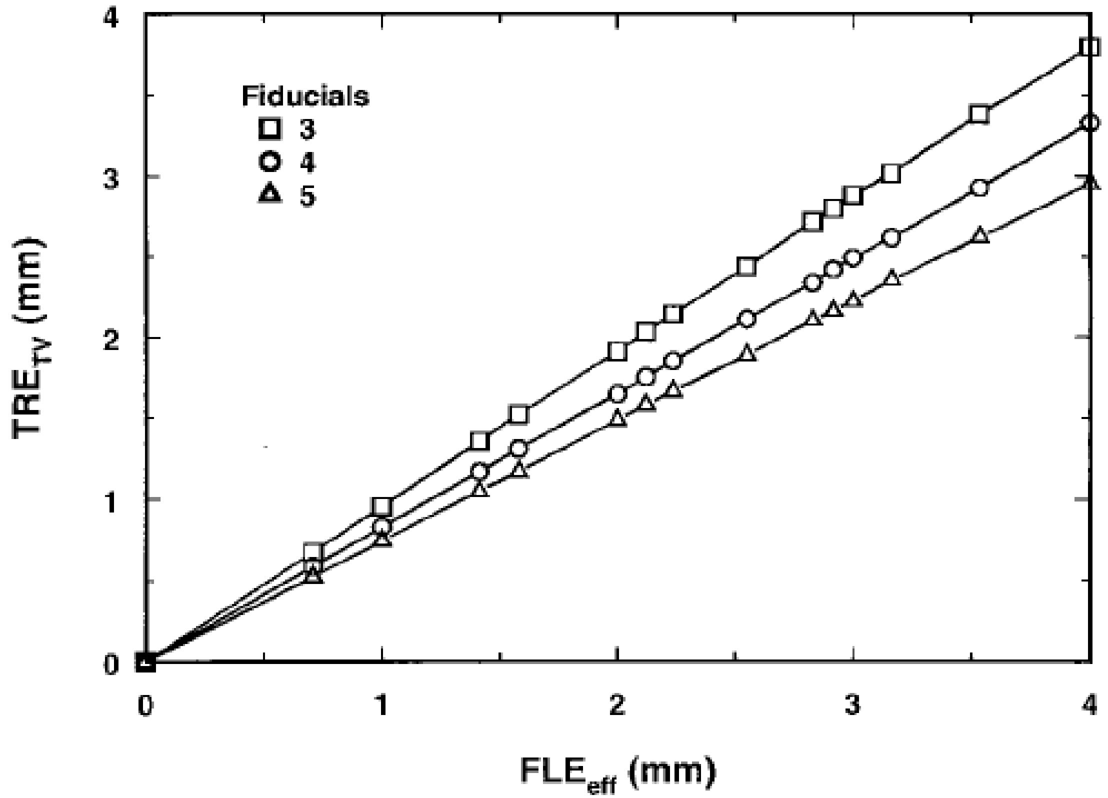


Fig. 8. Relationship between TRE and FLE. This figure illustrates that TRE is proportional to FLE$_{eff}$. This is shown for the cases of three, four, and five fiducials distributed evenly around the circumference of a sphere of radius 100 mm. Each symbol represents the mean theoretical TRE$_{TV}$ predicted by numerical simulation using a pair of FLE$_1$ and FLE$_2$ values formed from the set of localization errors {0, 1, 2, 3, and 4 mm}. All 15 possible pairs were used.

Horn's Method and TRE

- simulation of TRE vs number of markers

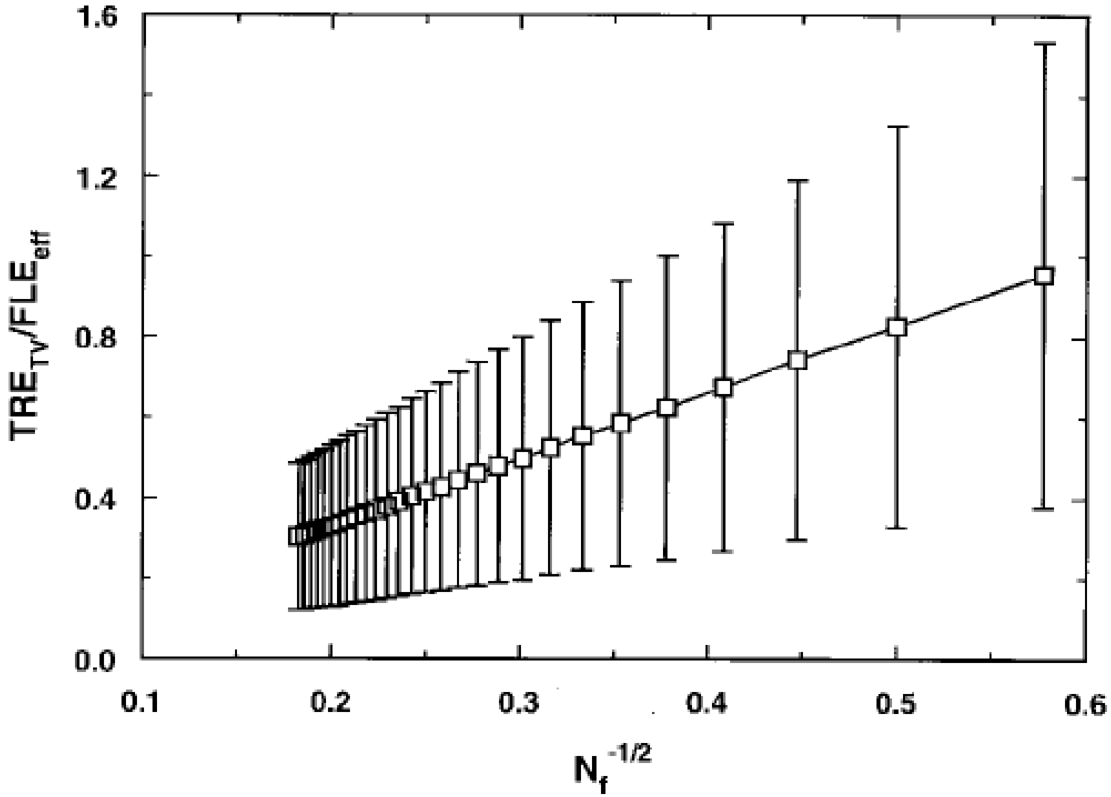


Fig. 9. Relationship between TRE and the number of fiducials. This figure illustrates that TRE is inversely proportional to the square root of the number of fiducials N_f. The symbols and error bars represent theoretical TRE_{TV}/FLE_{eff} values (mean $\pm$ SD) predicted by numerical simulation. The number of fiducials varies from three (far right) to 30 (far left). The fiducials were distributed evenly around the circumference of a sphere of radius 100 mm.

Horn's Method and TRE

- summary of results:
  - TRE depends strongly on configuration of markers
  - $\text{TRE}^2 \quad \propto \quad \sigma^2$
  - $\text{TRE}^2 \quad \propto \quad \dfrac{1}{n}$

Analysis of TRE

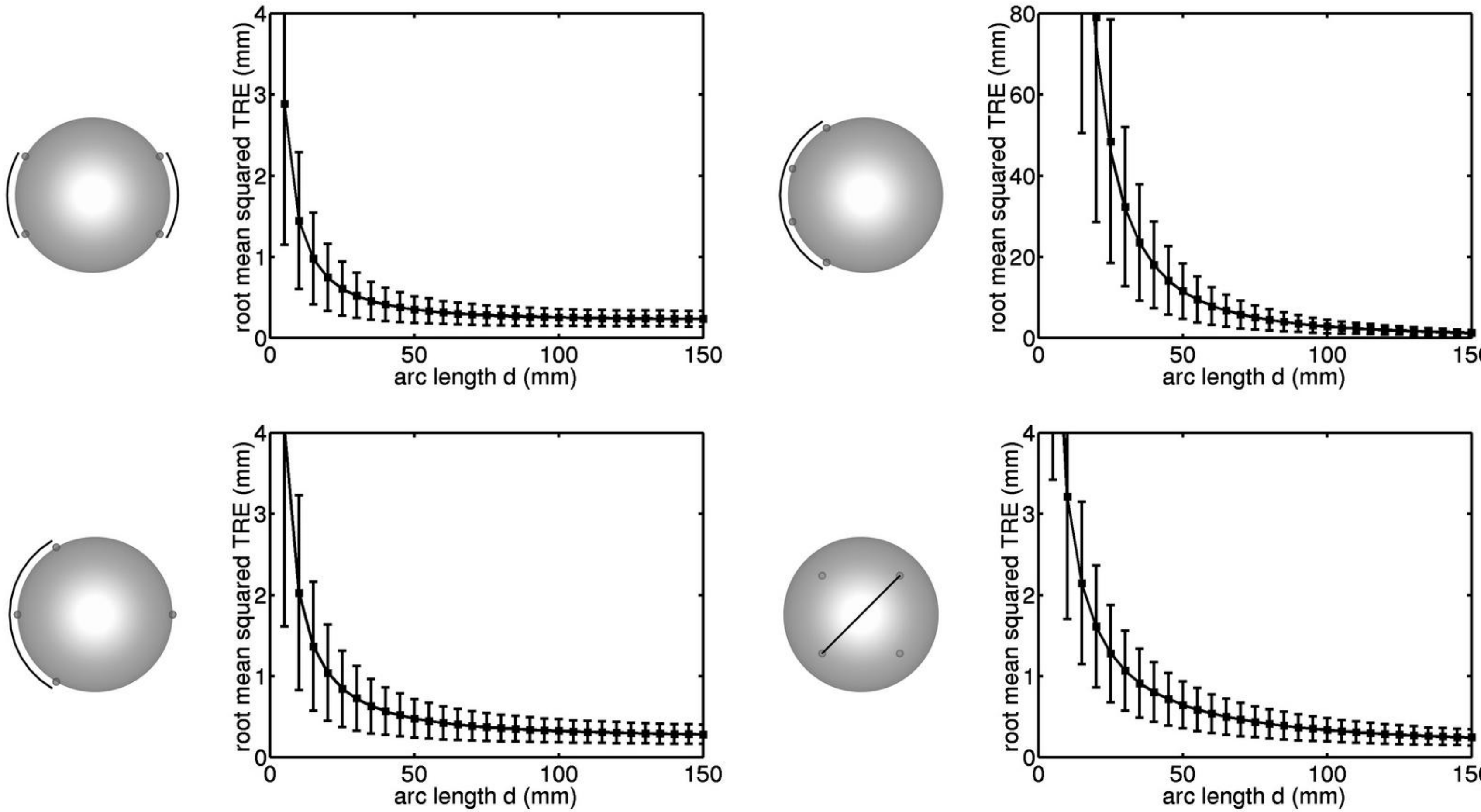
- Fitzpatrick, West, and Maurer Jr performed a statistical analysis of fiducial registration (assuming identical isotropic FLE)
  - IEEE Transactions on Medical Imaging, vol. 17, no. 5, Oct 1998

$$\langle \mathrm{TRE}^2(t) \rangle = \frac{\langle \mathrm{FLE}^2 \rangle}{n} \left(1 + \frac{1}{3} \sum_{k=1}^{3} \frac{d_k^2}{f_k^2} \right)$$

- t target location
- n number of markers
- d_k distance between the target and the k th principal axis
- f_k RMS distance between the fiducials and the k th principal axis

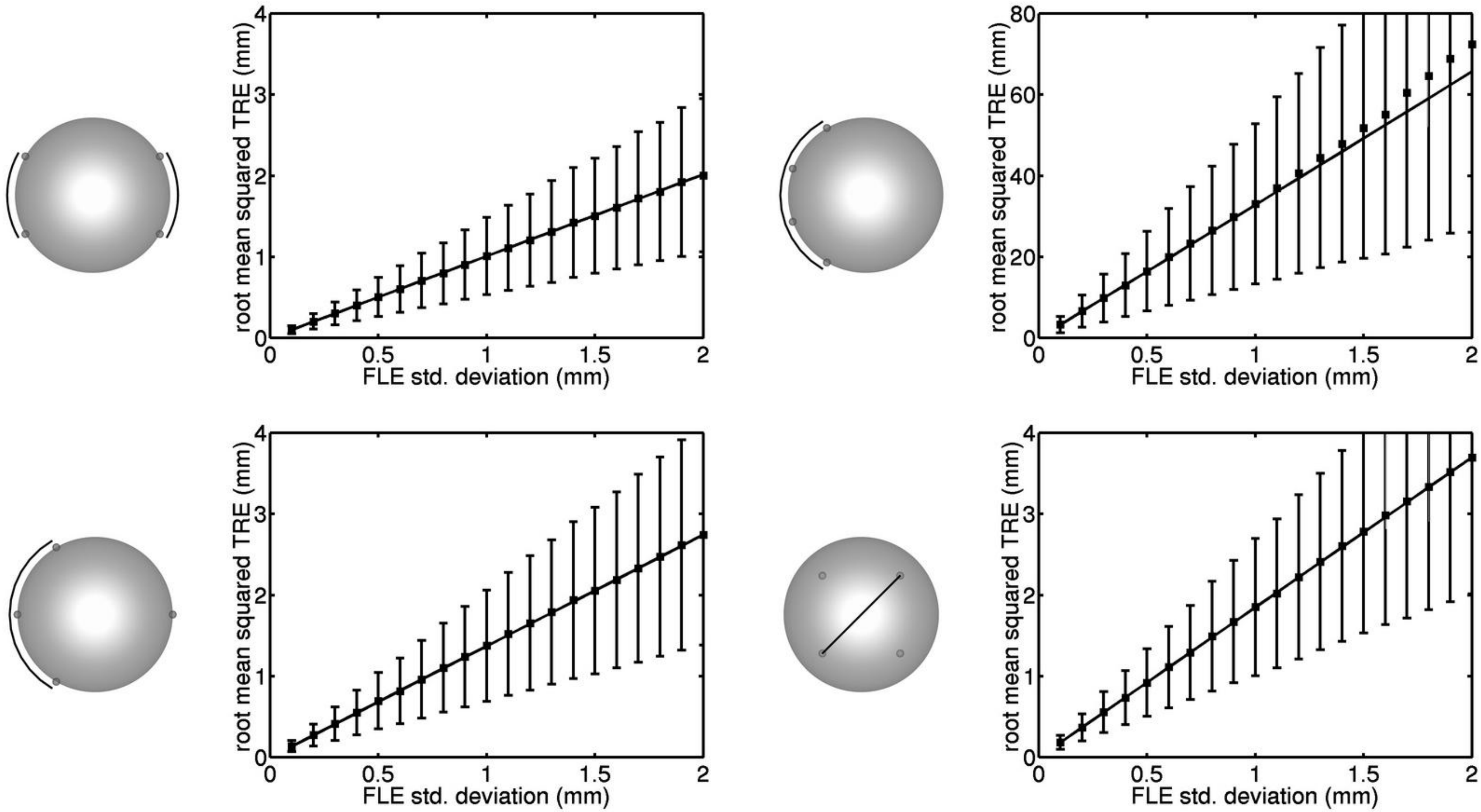
Analysis of TRE

- TRE for different configurations of markers



Analysis of TRE

- TRE versus FLE



Analysis of TRE

- other interesting results
  - FRE is independent of the marker configuration!

$$\langle \mathrm{FRE}^2 \rangle = \left(1 - \frac{2}{n}\right) \langle \mathrm{FLE}^2 \rangle$$

  - if you compute the TRE for the fiducial markers you get

$$\langle \mathrm{FRE}_i^2 \rangle = \langle \mathrm{FLE}^2 \rangle - \langle \mathrm{TRE}^2(p_i) \rangle$$

    - i.e., a small FRE implies a large TRE at the marker location!
    - FRE is poor indicator of registration quality